A Nonimaginary Approach to Complex Numbers

Catherine Miller, Olena Ostapyuk and Doug Shaw

The poet W. H. Auden, writing about the sad state of his mathematical education, recalled the mnemonic he was taught,

Minus times minus equals plus:
The reason for this we need not discuss.

We all recognize that "because I said so" is not an effective teaching method. But, can *you* explain why the product of two negative numbers is positive?

Here's a series of questions a math educator asked University of Northern Iowa students: It is zero degrees out. The temperature is decreasing two degrees per hour. What will the temperature be in three hours? "Negative six!" they replied. What was it three hours ago? After a pausing to think it through, they yelled "Six!" There we have it: Negative two times negative three equals positive six. And they understood, *really* understood.

What about complex numbers? Did you dutifully accept the square root of negative one when you first saw it, or did you resist? Most of us had no philosophical problems when we were introduced to vectors, functions, and matrices. But when Mr. Armstrong or Ms. Hansen expected us to accept a number, i, that created a negative number when squared, many of us cried foul. You can't square a number and get a positive result. We've taken the time to show that negative times negative equals positive. And positive times positive equals positive.

Our objections were not to complex numbers per se, but rather to taking the square root of a negative number. We'd forgive you for thinking you can't have one without the other. But, as we'll see, engineers use complex numbers all the time, and they don't resort to the obscenity of taking square roots of negative numbers.

The Standard Justifications

Textbooks and teachers define i by the relation $i^2 = -1$. (They don't define i as *the* square root of –1 because in the complex number system, –1 has two square roots, i and $-i$.) Then they add i to the set of real numbers and use a touch of algebra to get the complex numbers.

There are two ways they made rational sense of i. Yes, two.

Pragmatic: "Complex numbers are used in studying wave motion in electric circuits" (Connally, Hughes-Hallett, Gleason, et al. *Functions Modeling Change: A Preparation for Calculus*). Or, "We need to be able to completely factor polynomials." In other words, "complex numbers are useful, so we study them."

Theoretical: "Extending the real numbers to include a solution to the equation $x^2 = -1$ is analogous to extending the integers to include a solution to the equation $2x = 1$" (Cohen, Lee, and Sklar, *Precalculus*). In short, "because I said so."

There are problems with both approaches.

Pragmatic: Are complex numbers useful in the real world? Yes, but so is the tooth fairy—as a way of getting children to be happy about the scary prospect of losing a tooth. While the pragmatic approach is an honest one, it isn't a satisfying approach for many students.

Theoretical: Do complex numbers allow us to solve otherwise unsolvable equations? Sure, but mathematics is lousy with unsolvable equations such as $0x = 5$ and $y = \lim_{x\to 0}(|x| / x)$, and we don't violate basic mathematical laws to invent solutions to them.

It is *good* to bristle at the existence of i. If a teacher writes $\sqrt{x+16}=\sqrt{x}+4$ on the whiteboard, is it the good student or the poor student who nods, smiles, and silently copies the mistake into a notebook? If you complained that i was a silly thing because it didn't exist, you were right. So, let's do complex numbers without it.

The Way of the Engineer

Two boxers are on either side of you intending to use your head as a punching bag, repeatedly punching with a nice, even frequency. If each boxer is throwing 30 punches per minute, how many punches hit your head in 60 seconds? It depends. If they throw their punches simultaneously, it is going to hurt, because $30+30=60$. However, if one fist is advancing while the other is retreating, then by weaving back and forth, you can avoid all the punches, and $30+30=0$ (see figure 1).

As a more serious example, what do we get when we add a cosine curve with amplitude 1 to one with amplitude 2? A cosine wave of amplitude $1+2=3$, if they are in sync, as in figure 2.

But if they are not in sync, there are other possibilities: $1+2=-1$, $1+2=\sqrt{5}$, or any number in between -1 and 3, as shown in figure 3. In the context of adding cosine waves (which is done all the time in signal processing and electrical engineering), we can't just say $1+2=3$.

Just as vectors are quantities with a magnitude and a direction, we can think of sinusoidal functions with a fixed frequency as quantities with a *magnitude* and a *phase*. We call such quantities *phasors* and denote them (magnitude) $\angle$ (phase).

Figures 2 and 3 show the addition of pairs of cosine curves that are completely in phase, 180 degrees out of phase, and 90 degrees out of phase. The pair in figure 3b have amplitudes 1 and 2 and are 90 degrees out of phase. The sum has amplitude $\sqrt{5}$ and is about 63 degrees out of phase. In our new notation we would write: $1\angle 0^\circ + 2\angle 90^\circ \approx \sqrt{5}\angle 63^\circ$.

As we shall see, phasors are the complex numbers in disguise. But we have defined them without doing anything ridiculous or objectionable. We can't hold $2\angle 90^\circ$ apples in our hand. But we can't hold (0,2) apples, $\begin{pmatrix}0\\2\end{pmatrix}$ apples, $\begin{pmatrix}3&-2\\1&4\end{pmatrix}$ apples, or x^2-3x+2 apples

Figure 2. $1+2=3$.

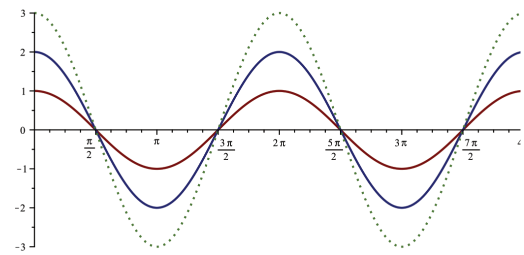


Figure 3. (a) $1+2=-1$. (b) $1+2=\sqrt{5}$.

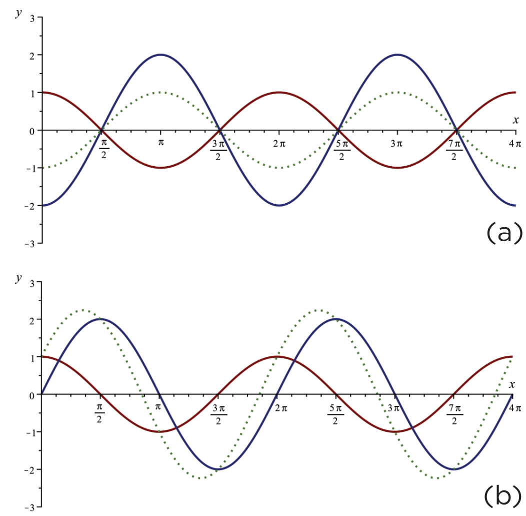
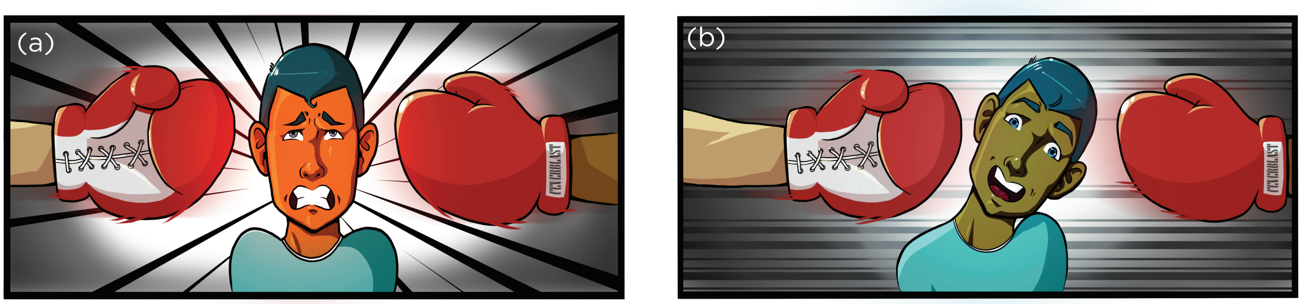


Figure 1. (a) $30+30=60$. (b) $30+30=0$.



(credit: Ray Smith)

either. Phasors do not represent a number of apples.

Sums of Phasors

As our previous examples show, the sum of two sinusoidal functions with the same frequency is another sinusoidal function of that frequency. But it is messy to compute the phase and magnitude of the sum. Fortunately, it is easy to compute the sum if we view phasors as vectors. In fact, phasor is a portmanteau of "phase vector."

We can represent a phasor $r\angle\theta$ as a vector rooted at the origin: r is the length of the vector, and θ is the angle the vector makes with the positive x-axis (see figure 4). If the tip of the vector is at (x,y), we write this phasor as $x + yj$. Note that engineers use j to specify the y-coordinate because they use i for current (from the French *intensité de courant*) and because j corresponds to the unit vector $\mathbf{j}$.

We add phasors just as we add vectors. (We omit the proof that this sum of vectors is equivalent to the sum of the sinusoidal functions.) To compute the sum in figure 3b, we convert the phasors to $x + yj$ notation, add, and then convert back to (magnitude) $\angle$ (phase) notation:

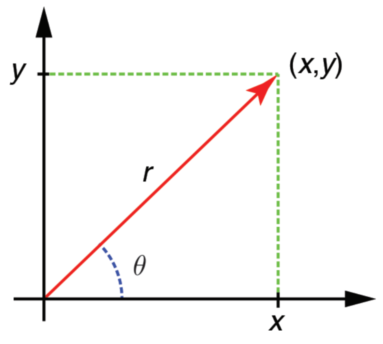
$$\begin{aligned} 1\angle 0° + 2\angle 90° &= (1 + 0j) + (0 + 2j) \\ &= 1 + 2j \\ &= \sqrt{5}\angle \arctan(2) \\ &\approx \sqrt{5}\angle 63°. \end{aligned}$$

Products of Phasors

To multiply phasors, we multiply the magnitudes and add the phases:

$$(a_1\angle\theta_1)(a_2\angle\theta_2) = a_1a_2\angle(\theta_1 + \theta_2).$$

Figure 4. The phasor $r\angle\theta = x + yj$.



This definition generalizes the multiplication of real numbers. For instance,

$$-2 \cdot 3 = (2\angle 180°)(3\angle 0°) = 6\angle 180° = -6$$

and

$$(-2)(-3) = (2\angle 180°)(3\angle 180°) = 6\angle 0° = -6.$$

The reader can verify that phasor addition and multiplication satisfy all the requirements of a field: They satisfy the distributive property, every nonzero phasor has a multiplicative inverse, and so on.

This definition also gives multiplication a geometric significance. We can think of taking the first phasor, multiplying its length by a_2, and then rotating it by θ_2. For example, a double rotation by 90 degrees is a rotation by 180 degrees. So, $(1\angle 90°)(1\angle 90°) = 1\angle 180°$, or in other notation, $j^2 = -1$! Consequently, phasor arithmetic is just complex arithmetic in disguise. Thus, we can define our $x + yj$ as a "complex number," and we can do anything we want with them.

But the main idea is this: $j^2 = -1$ is *not* a definition. It is a notational accident! The statement is just saying that when we take a cosine wave that is 90 degrees out of phase with a reference wave, and we shift it another 90 degrees out of phase, we get one that is 180 degrees out of phase. In other words, if you make two quarter-turns, the net result is a half turn.

What's not to like?

***Catherine Miller**, a mathematics professor at the University of Northern Iowa, works to remind secondary teachers about their love of mathematics and researches social justice in mathematics education.*

***Olena Ostapyuk** is an assistant professor of mathematics at the University of Northern Iowa. Her research interests include complex dynamics, geometric function theory, and innovative mathematics pedagogy.*

***Doug Shaw** is a mathematics professor at the University of Northern Iowa, who once had a professor, Dale Munson, who explained complex numbers to him and thousands of electrical engineering students as was done here, minus the punching. He also had a history teacher, Ernest Lidge, who taught him and thousands of high school students to think like mathematicians.*

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