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Harris Graphs—A Graph Theory Activity for Students and Their Instructors

Douglas J. Shaw

Doug Shaw (doug.shaw@uni.edu; MR ID 803711) is a professor of mathematics at the University of Northern Iowa. He is proud to have taught in the Michigan Math and Science Scholars program at the University of Michigan, where the events described in this article occurred. He conducts professional development workshops for educators using applied improvisation techniques to improve teaching. As he is typing this, he is making eye contact with a dog whose head has been dyed purple because his family foolishly allowed him to take it to the groomer unsupervised.

Graph theory is so much fun to teach! How often do you get a group of students on equal footing, unaffected by how many times they had to take calculus? If they have developed the ability to think and solve problems, with some creativity intact, then your students are starting as peers, ready to go!

Unfortunately, most problems for new graph theorists fall into three categories.

(1) Application of definitions: "Prove the following two graphs are isomorphic." "Draw all order 5 trees." These are good problems to begin a problem set, but they bore the advanced students and are often uninteresting to the rest.

(2) Proofs: "Prove that if a graph is connected and even degreed, then it is Eulerian." "Prove that if a graph has no odd cycles, then it is bipartite." These are good problems for mathematically mature students and are great at the end of a set, but there is a limit to how many proofs students are ready to tackle, particularly those who have never had an abstract mathematics class. Alternatively, some proofs, while instructive, almost fit into the previous category, e.g., "Prove that if $G \cong H$ and $H \cong I$, then $G \cong I$."

(3) Finding counterexamples: "Find a graph with chromatic number 4 that does not contain a K_4." These problems are far too rare in textbooks. They allow students to explore and play with concepts, and the instructor can choose to require the level of justification required. Unfortunately, most of these problems are either relatively simple or extremely complex.

This is a story of how an inquiry-based classroom atmosphere led to the discovery of a graph theory problem challenging enough that a group of students will not be able to solve it in fifteen minutes, but not so challenging that it cannot be done in an hour. Most importantly, it relies on the abilities to "play" and "try something instead of nothing" that are part of both the inquiry-based learning (IBL) process [1] and professional mathematics research. It makes a great in-class activity as checking to see if a proposed solution is correct is doable but nontrivial. So while the more impulsive students are at the blackboard, drawing graphs, modifying graphs, and yelling at each

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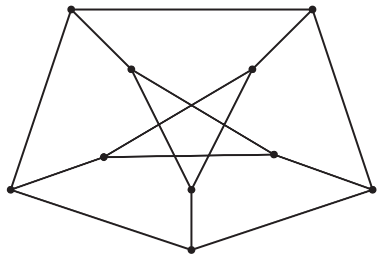

Figure 1. The Petersen graph.

other, the more methodical ones are working on proposed solutions, trying to falsify them, and also probably yelling. Your job, as the instructor, will be to show excitement (which you will not have to feign) and say things like, "Hey! Janice and Joe say they got one! I don't see anything immediately wrong with it! We need some people to check!" Teamwork will arise naturally, and you may be surprised at which students wind up finding a correct solution.

Definitions

We assume some familiarity with graph theory; see [**2**] for more details. Recall that a graph is Eulerian if it is connected and all of its vertices have an even number of neighbors. (There is a more common definition of Eulerian, but this is the one we will use.) And a graph is Hamiltonian if it is possible to travel from vertex to vertex, along the edges, using each vertex exactly once and ending up where we started. (Arguably, the first vertex does get used twice, once at the beginning, and once at the end). We will also need the important Petersen graph (Figure 1).

There is a necessary condition for a graph to be Hamiltonian that I call the $G \setminus S$ condition (I know of no standard name for this property). For a subset S of the vertices of G, let $G \setminus S$ be the graph obtained by removing the vertices of S along with their associated edges.

Definition. *A graph G has the $G \setminus S$ condition if, for every subset S of the vertices of G, the number of components of $G \setminus S$ is at most the number of vertices in S.*

Figure 2 shows a graph that does not have the $G \setminus S$ condition: If S is the three circled vertices, then $G \setminus S$ has four components. Therefore we know right away that G cannot be Hamiltonian.

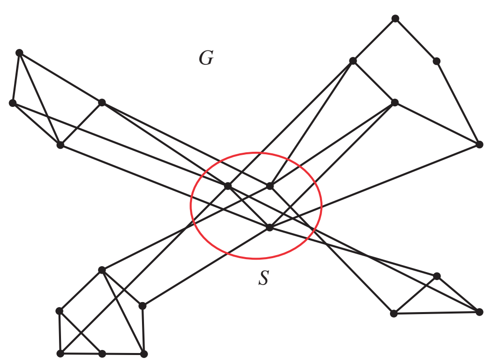

Figure 2. A graph without the $G \setminus S$ condition.

The tale

During a discussion of various necessary and sufficient conditions for a graph to be Hamiltonian, one student, Harris Spungen, asked why the $G \setminus S$ condition was not a sufficient condition. I put the Petersen graph on the board; it can be shown to have the $G \setminus S$ condition but not be Hamiltonian.

A few minutes later, Harris raised his hand and said that he had come up with a new sufficient condition:

Conjecture (Harris). *If a graph has the $G \setminus S$ condition and is Eulerian, then it is Hamiltonian.*

I believed his conjecture had to be false, because if it were true, it would be a standard sufficient condition for Hamiltonian graphs. So I said, "I don't believe you are correct," and tried to think of a quick counter-example, but could not. "I'll get back to you on this tomorrow," I said, and went back to my lecture plan, forgetting everything I had learned about IBL or good teaching in general. Then, in the middle of my next sentence, I thought, "Wait! What am I doing?!" and said, "Hey—you know what? Nothing else I had planned for today is as interesting as the Harris conjecture. Let's try to find a counter-example."

Here was the soul of IBL: A genuine, student-created question came up organically, and now the class explores the question, attempting to find the answer.

Some students went to the chalkboard immediately, drawing and arguing. Others were hanging back. And then, when students in the former group were saying "'I think I got it," instead of checking their work myself, I got the seated people involved. "Can you see if they are right?" Ten minutes passed. Twenty. Thirty. When the checkers were convinced that a counterexample worked, a teaching assistant and I would take a look, dashing hopes to the ground. But we were all, everyone in the room, excited. Had Harris come up with a new condition after all?

Finally, after about 45 minutes, one graph seemed to work. So I asked the entire class to help confirm. All 20 people in the room gathered around, students, teaching assistants, myself. Looking at all possible sets S, deleting them mentally, and counting components. Starting with one vertex, moving to pairs, then to triples, verifying the $G \setminus S$ condition. Then convincing ourselves that no Hamiltonian cycle existed.

I said to the class, "This graph is called the Harris graph!" My error was using the word "the." As the years went on, other classes tackled the problem and the bestiary of Harris graphs grew. An appendix at the end of this article lists all the Harris graphs we have found so far. Before looking at that page, take out a sheet of paper, or a magna-doodle, and draw your first Eulerian, non-Hamiltonian graph that has the $G \setminus S$ condition. (Added in page proofs: In the Summer 2018 session, Hirotaka Yoneda (student) found an order 7 Harris graph and we have proven this is the minimal order. It is not included in the appendix—maybe your students can find it!)

Conclusion

Finding Harris graphs is a nontrivial activity that is inherently group-oriented. The process of looking for them gives students an experiential feel for non-Hamiltonian graphs and how tough it is to create one (without the easy out of having it fail the $G \setminus S$ condition). The students are not talking about mathematics; they are doing mathematics, and working on a problem that does not have an answer that can be found on the internet (until now). In addition to the satisfaction a class experiences upon discovering a

Harris graph, students also learn about the value of resilience or “grit,” which has been shown to be a key factor in collegiate success in STEM fields [3].

As instructors, we can discover many interesting questions and activities by being willing to replace the phrases “I don’t know” and “I’ll get back to you on that” with “That’s an interesting question. Let’s see what we can figure out together.” Try it, and see where you wind up!

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Summary. We define Harris graphs and tell the tale of their discovery. A class can usually find a Harris graph in an hour, but usually not in fifteen minutes, making them an ideal way to teach both graph theory and mathematical perseverance. Their discovery illustrates how inquiry-based learning does not have to be premeditated and can transform the course of a lesson.

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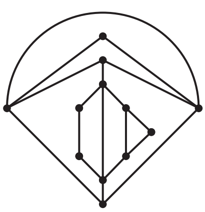
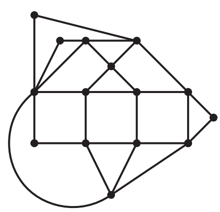
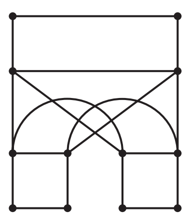
Appendix: Harris graphs (curated by Elizabeth Petrie)



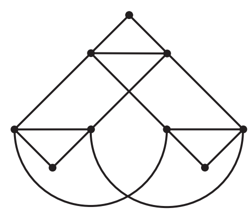
Jayna Fishman and Elizabeth Petrie (students)



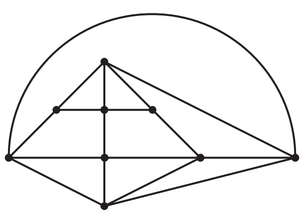
Jungmin Kwon, Sunny (Jingheng) Li, Minseo Son, Ruihan Wang (students)



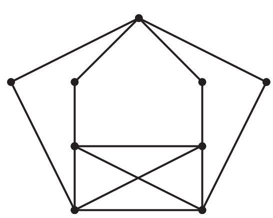
Zack Hylaird and Joseph Stafford (students)



(simplification of previous graph) Alex Leaf (teaching assistant)



Jayna Fishman and Elizabeth Petrie (students)



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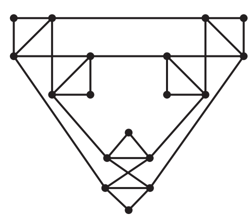

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